A two-dimensional air intake in a sonic stream

By L. E. FRAENKEL
Aeronautics Department, Imperial College, London

(*Received* 20 *March* 1958)

SUMMARY

In this paper transonic small-disturbance theory is applied to a simplified model of the flow near the front of a ducted body. The body is assumed to consist simply of two parallel flat plates which extend from the inlet station to infinity downstream. The velocity far upstream is sonic, and the velocity far downstream in the duct, which is assumed to be known, is slightly subsonic. Air is therefore 'spilled' around the intake edges. An analytic solution is found for the resulting flow field up to the 'limiting Mach wave', and asymptotic solutions are found for the supersonic flow and for the shock wave far from, and near, the intake edges. The pressure distribution along the outside walls is then known at both ends, and its computation is completed by an empirical procedure. Distributions of pressure along the centre-line and along the inside and outside walls are shown. These results may be used to compute the drag of sharp-edged intakes with a very small frontal area.

1. INTRODUCTION

Under most flight conditions, the intakes of high-speed aircraft 'spill' air: that is, the mass-flow requirement of the engine is such that the entering stream-tube of air has a cross-sectional area, far upstream, which is smaller than the area of the hole at the front of the body. As a result, air is diverted around the intake edge, and at high subsonic and at all supersonic flight speeds the flow field is of mixed type (see figure 1). In the subsonic case there are regions of supersonic flow near the edge and, possibly, further back along the external wall of the intake. In the supersonic case there is a subsonic region which is bounded by the bow shock wave, by a sonic surface from this shock to the intake edge, and by the walls of the duct. As far as the external flow is concerned, the outer wall of the intake acts somewhat like the upper surface of an aerofoil, increasing spillage corresponding roughly to increasing incidence. These mixed intake flows have been the subject of considerable experimental work, and various empirical theories have been developed to predict the effect of spillage on the drag of aircraft, but (as far as the writer is aware) nothing approaching a theoretical description of the flow field has been developed.

In the present paper we study the simplest possible example of such a flow. The intake is assumed to be two-dimensional and to consist simply of two parallel, semi-infinite, flat plates. Alternatively, the upper half of this configuration may be interpreted as a simplified model of one of the side intakes found on some aircraft. The final velocity in the duct (which in practice is known from the air mass flow of the engine) is assumed to be only slightly less than that in the free stream, so that perturbations from a uniform stream are small, except in the neighbourhood of the intake edge. If the free-stream Mach number is near one, transonic small-disturbance theory may then be applied, and the central problem, in which the free stream is sonic, is considered here.

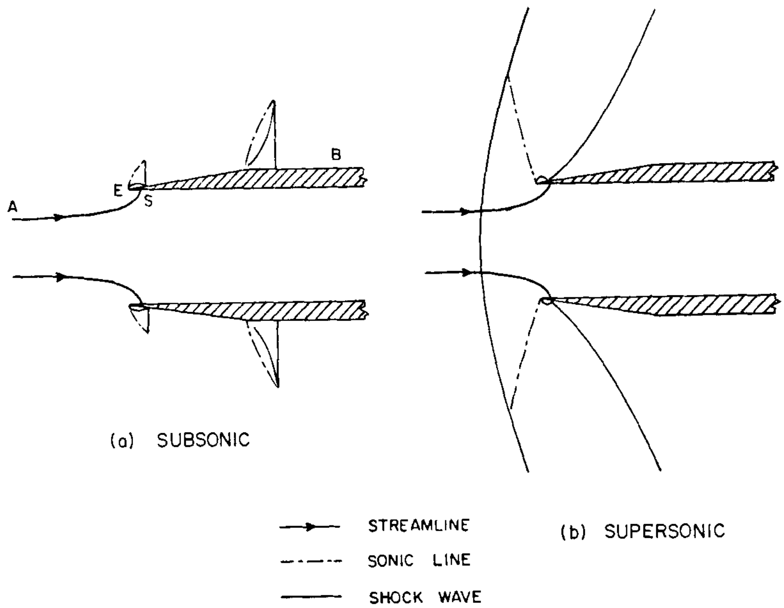

Figure 1.

It is usual in intake work to use a definition of drag which for plane flow becomes

$$D = 2 \int (P - P_0)\, dY,$$

where P_0 denotes the pressure in the free stream, and the integral is taken over the entire length of the dividing streamline, $ASEB$ in figure 1. This definition is consistent with the usual definition of engine thrust. It is natural to consider the drag as the sum of two parts: (i) the 'pre-entry drag', which is the contribution of the upstream part, AS, of the dividing streamline, and can be calculated from the known change in momentum and pressure of the internal flow; and (ii) the 'cowl drag', which in the case of a flat-plate intake is simply the force on the edge E. In incompressible flow theory these two drags precisely cancel each other, as might be expected, regardless of whether exact or small-disturbance theory is used. In the present transonic problem, however, the singularity in velocity at the edge is rather weak, and the edge force is found to be zero. Both in incompressible and in transonic flow, the edge flow has, of course, the same character as that at the leading edge of a lifting flat plate, for which the transonic solution has been found by Guderley (1954). Hence the result of zero edge force, which was not discussed by Guderley, applies to that problem also.

Some further information about cowl drag can also be obtained from the present model. Consider a wedge intake (that is, a two-dimensional intake of the type shown in figure 1) in a sonic stream. For moderate wedge angles the flow up to the 'limiting Mach wave' (figure 2 (*a*)) is the same as that past a flat-plate intake. (The significance of the limiting Mach wave will become clear from what follows.) Further, if the wedge angle is of even smaller order than the slope of the dividing streamline ahead of the intake, then the dominant part of the pressure acting on the wedge face is that on the outer wall of the flat-plate intake. Hence for such slender-wedge intakes a first approximation to the (negative) cowl drag can be found from the present solution.

2. The equations of transonic flow

Let X, Y be Cartesian coordinates such that a free stream of near-sonic velocity U_0 and of Mach number M_0 flows in the direction of increasing X. Let the velocity at a point in the field be $(U_0 + U_0 U', U_0 V')$. The fluid is assumed to be a perfect gas whose specific-heat ratio is γ. Then for small disturbances the equations of transonic flow may be written

$$\left.\begin{aligned} (1-M_0^2-\Gamma U')U'_X + V'_Y &= 0, \qquad (\Gamma = (\gamma+1)M_0^2), \\ U'_Y - V'_X &= 0, \end{aligned}\right\} \tag{2.1}$$

where subscripts denote partial derivatives, and where we have adopted Spreiter's (1954) version of these equations. In order to work with variables whose magnitude is $O(1)$, and to express the equations in canonical form, we introduce a small parameter δ, such that $V' \sim O(\delta)$, and a length L, representative of the lateral extent of the field, and make the transformation

$$\left.\begin{aligned} X &= L(\delta\Gamma)^{1/3}x, & Y &= Ly, \\ U' &= (1-M_0^2)\Gamma^{-1} + \delta^{2/3}\Gamma^{-1/3}u(x,y), & V' &= \delta v(x,y). \end{aligned}\right\} \tag{2.2}$$

Then equations (2.1) become

$$\left.\begin{aligned} -uu_x + v_y &= 0, \\ u_y - v_x &= 0, \end{aligned}\right\} \tag{2.3}$$

with the boundary condition

$$u \to (M_0^2-1)(\delta\Gamma)^{-2/3} = K, \quad v \to 0, \quad \text{as } x, y \to \infty.$$

Here K is the transonic similarity parameter.

If the roles of the independent and dependent variables are now interchanged, equations (2.3) become

$$x_u - u y_v = 0,$$
$$x_v - y_u = 0,$$

so that the Legendre contact potential, $\chi = \int (x\,du + y\,dv)$, satisfies the Tricomi equation

$$\chi_{uu} - u\chi_{vv} = 0. \tag{2.4}$$

The characteristic curves of this equation are $v = \frac{2}{3}u^{3/2} + \text{constant}$, $(u > 0)$: hence we introduce the variables $\zeta = \frac{2}{3}u^{3/2}$ in the hyperbolic half-plane, and $w = -u$, $z = \frac{2}{3}w^{3/2}$ in the elliptic half-plane. Then the equation becomes

$$\chi_{\zeta\zeta} + \frac{1}{3\zeta}\chi_\zeta - \chi_{vv} = 0, \qquad \chi_{zz} + \frac{1}{3z}\chi_z + \chi_{vv} = 0. \tag{2.5}$$

Two types of elementary solution of this equation, and the relations between them, are discussed in Appendix I.

To the lowest order, the stream function is $\Psi = \int \rho_0 U_0\, dY$, so that $y\ (= \chi_v)$ may be taken constant on streamlines. The variable y also satisfies the Tricomi equation.

3. Solution of the intake problem up to the limiting Mach wave

Figure 2 shows qualitatively the physical and hodograph planes of the present problem. The origin of coordinates is taken at the intake edge, and the length L in (2.2) is taken as half the distance between the two flat plates. The final velocity in the duct being U_{I}, we define δ by

$$\delta^{2/3}\Gamma^{-1/3} = \frac{U_0 - U_{\mathrm{I}}}{U_0}. \tag{3.1}$$

By continuity, the overall change in width of the entering streamtube is then $O(\delta^{4/3})$, and this occurs essentially over an X-length of $O(\delta^{1/3})$, so that the slope of the dividing streamline is $O(\delta)$.

The physical and hodograph fields may be described as follows. The origin in the physical plane maps to infinity in the hodograph plane, and vice versa. The most important streamline is the dividing streamline, $y = 0$. As this approaches the stagnation point, S, it has positive slope $(v > 0)$ and the velocity is reduced $(w > 0)$. At S $(x = 0, y = 0, w = \infty$, $v = 0$ on the approximate theory) the dividing streamline branches; one part, $y = 0-$, running along the inner wall SI, while the other, $y = 0+$, expands at infinite velocity about the edge E. In fact it overexpands, as do all the external streamlines, and ultimately attains negative slope $(v < 0)$. Recompression and the return to zero slope occur first through a shock, and then continuously. The shock corresponds to a jump in position in the hodograph plane, along the appropriate shock polar, and the subsequent flow maps on to a second sheet of the hodograph, which is here shown superposed on the first.

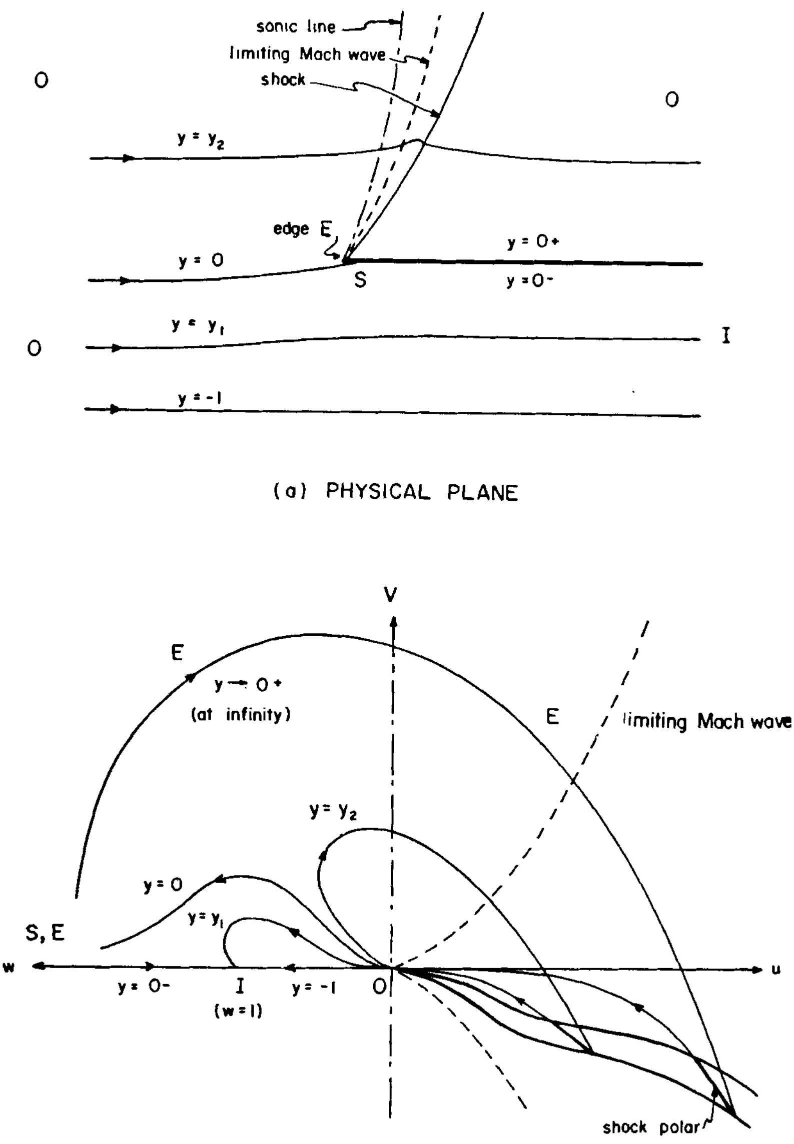

Figure 2.

The central streamline of the intake, $y = -1$, proceeds from O to I at zero slope. The internal flow is contained between $y = 0$, $y = 0-$, and $y = -1$; the external flow between $y = 0$, $y = 0+$, and the shock. The limiting Mach wave is the last of the expansion family emanating from the edge to meet the sonic line; hence disturbances downstream of this line do not affect the subsonic field. It maps on to $v = \zeta$.

In this section we consider only the flow up to the limiting Mach wave, that is, the problem in $\mathscr{R}(v-\zeta) \geqslant 0$. This is essentially a boundary value problem of the Tricomi type. The solution is constructed in two parts as follows ($\chi = \chi_1 + \chi_2$).

(i) The potential χ_1 is to represent the far field of the physical plane, that is, the flow near O. We require:

$$\left.\begin{aligned} &y_1 = \chi_{1v} \to \infty \ \text{ as } \ u, v \to 0, \\ &y_1 \to 0 \ \text{ as } \ u, v \to \infty, \\ &y_1 = 0 \ \text{ on } \ v = 0, \ w > 0, \\ &y_1 \text{ finite on } v = \zeta, \ v > 0, \end{aligned}\right\} \tag{3.2}$$

and a one-to-one mapping from the hodograph to the physical plane. This 'free stream singularity' for any symmetrical body has been found by Frankl (1947) and Guderley (1948). It is one of the similarity solutions $\chi = \zeta^{-m} fn(\alpha)$, where $\alpha = v^2/\zeta^2$ (Appendix I). In the present case $m = \frac{2}{3}$, and the $fn(\alpha)$, which is a hypergeometric function, reduces to elementary form. In fact

$$\chi_1 = -\mu \frac{\sqrt{3}}{\pi} \zeta^{1/3} \int_0^\infty e^{-\lambda v} K_{1/3}(\lambda\zeta)\, d\lambda, \tag{3.3 a}$$

$$= -\mu 2 z^{-2/3} \frac{\cosh\{\frac{1}{3}\cosh^{-1}\sqrt{(1-\alpha)}\}}{\sqrt{(1-\alpha)}}, \qquad (v \geqslant 0, \quad \alpha \leqslant 0), \tag{3.3 b}$$

$$= -\mu 2 \zeta^{-2/3} \frac{\sinh\{\frac{1}{3}\sinh^{-1}\sqrt{(\alpha-1)}\}}{\sqrt{(\alpha-1)}}, \qquad (v \geqslant 0, \quad \alpha \geqslant 1), \tag{3.3 c}$$

where μ is a constant to be determined in any particular problem. Although it is irrelevant at this stage, we also note that

$$\chi_1 = -\mu 2 \zeta^{-2/3} \frac{\sin\{\frac{1}{3}\sin^{-1}\sqrt{(1-\alpha)}\}}{\sqrt{(1-\alpha)}}, \qquad (0 \leqslant \alpha \leqslant 1), \tag{3.3 d}$$

where $\sin^{-1}\sqrt{(1-\alpha)}$ varies from 0 to π as v varies from ζ to $-\zeta$.

To formulate precisely the problem for χ_2, we observe that χ_1 gives velocities near the edge E which are altogether too large, and must be removed by χ_2. Indeed if χ_1 were to be the dominant term of the complete solution near E, this would result in an infinite drag, as will be shown in § 6.

(ii) The potential χ_2 is introduced to satisfy the boundary conditions on the body:

$$\left.\begin{aligned} y_2 &= -1 \ \text{ on } \ v = 0, \qquad 0 \leqslant w < 1, \\ &= 0 \ \text{ on } \ v = 0, \qquad\qquad w > 1, \\ y_2 + y_1 &\sim o(y_1) \ \text{ as } \ w, v \to \infty, \\ y_2 &\text{ is finite on } v = \zeta. \end{aligned}\right\} \tag{3.4}$$

Although μ is as yet unknown, the second of these conditions does fix the order of magnitude of y_2 at infinity in the hodograph plane. It is shown in Appendix II that (3.4) determine a unique boundary value problem in the sense of Tricomi.

We seek a solution of form

$$\chi_2 = -\frac{\sqrt{3}}{\pi}\zeta^{1/3}\int_0^\infty e^{-\lambda v}K_{1/3}(\lambda\zeta)f(\lambda)\frac{d\lambda}{\lambda}, \tag{3.5}$$

where the $I_{1/3}$-solution has been omitted because it leads, in general, to a singularity on the limiting Mach wave. (This step is justified rigorously by the fact that (3.5) will satisfy all the boundary conditions, and is therefore the unique solution.) $f(\lambda)$ is then determined from the integral equation

$$y_2(z,0) = z^{1/3}\int_0^\infty [J_{1/3}(\lambda z)+J_{-1/3}(\lambda z)]f(\lambda)\,d\lambda = -1, \quad \tfrac{2}{3} > z \geqslant 0,$$
$$= 0, \quad z > \tfrac{2}{3},$$

by means of the Mellin transform

$$\tilde{\phi}(p) = \int_0^\infty z^{p-1}\phi(z)\,dz$$

(see Appendix I for further details). The integral equation transforms to

$$[\tilde{J}_{1/3}(p+\tfrac{1}{3})+\tilde{J}_{-1/3}(p+\tfrac{1}{3})]\tilde{f}(-p+\tfrac{2}{3}) = -(\tfrac{2}{3})^p/p,$$

where (Erdélyi 1954)

$$\tilde{J}_{1/3}(p)+\tilde{J}_{-1/3}(p) = -\frac{\sin\frac{1}{2}\pi p\cos\frac{1}{6}\pi}{\cos\pi p-\cos\frac{1}{3}\pi}\frac{2^{p+1}}{\Gamma(-\frac{1}{2}p+\frac{5}{6})\Gamma(-\frac{1}{2}p+\frac{7}{6})}, \; (\tfrac{1}{3} < \mathscr{R}p < \tfrac{3}{2}). \tag{3.6}$$

Hence, with $-p+\frac{2}{3} = q$, we find after a little reduction that

$$\tilde{f}(q) = (\tfrac{2}{3})^{2/3-q}\frac{2^{q-3}}{\pi}\frac{\cos\pi q+\cos\frac{1}{3}\pi}{\cos\frac{1}{2}\pi q\cos\frac{1}{6}\pi}\Gamma(\tfrac{1}{2}q-\tfrac{1}{3})\Gamma(\tfrac{1}{2}q+\tfrac{1}{3}), \quad (|\mathscr{R}q| < 1), \tag{3.7a}$$

and

$$f(\lambda) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\lambda^{-q}\tilde{f}(q)\,dq, \quad (|c| < 1). \tag{3.7b}$$

The only singularities of $\tilde{f}(q)$ are simple poles at

$$q = \pm(2n+1), \qquad n = 0, 1, 2, \ldots,$$

the poles of the Γ-functions being removed by zeros of $\cos\pi q+\cos\frac{1}{3}\pi$. The evaluation of the contour integral in (3.7b) is therefore straightforward: we close the contour with a large semi-circle in $\mathscr{R}q < 0$, whose contribution vanishes in the limit for all finite λ, because of the strong convergence induced by the Γ-functions in (3.7a). Then

$$f(\lambda) = -\left(\frac{2}{3}\right)^{2/3}\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}(-)^n\left(\frac{\lambda}{3}\right)^{2n+1}\frac{1}{\Gamma(n+\frac{7}{6})\Gamma(n+\frac{11}{6})}, \tag{3.8a}$$

$$= -\left(\frac{2}{3}\right)^{2/3}\frac{2\sqrt{3}}{5\pi}\lambda+\left(\frac{2}{3}\right)^{2/3}\frac{8\sqrt{3}}{5.7.11.\pi}\lambda^3+\ldots. \tag{3.8b}$$

Now consider χ_2. If the substitution $\tilde{g} = -(\sqrt{3}/\pi)\tilde{f}$ is made in equation (I.8), and the resulting integral is evaluated by contour integration, χ_2 is given by various series of hypergeometric functions. However, we are primarily interested in $x(w,0)$, which gives the pressure distribution along the centre-line and the inner wall of our intake. Now on $v = 0$

the hypergeometric functions take on simple values, or alternatively (3.5), (3.6) and (3.7 a) may be used directly. In fact (3.5) may be written

$$\tilde{\chi}_2(-q-\tfrac{1}{3},0) = -[\tilde{J}_{1/3}(-q)+\tilde{J}_{-1/3}(-q)]\tilde{f}(q), \quad (-1 < \mathscr{R}q < -\tfrac{1}{3}),$$

so that, substituting from (3.6) and (3.7 a) and inverting, we have

$$\chi_2(z,0) = -\frac{1}{6}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{3}{2}z\right)^{q+1/3}\tan\tfrac{1}{2}\pi q\,\frac{\cos\pi q+\cos\frac{1}{3}\pi}{\cos\pi q-\cos\frac{1}{3}\pi}\times$$
$$\times\frac{\Gamma(\frac{1}{2}q-\frac{1}{3})\Gamma(\frac{1}{2}q+\frac{1}{3})}{\Gamma(\frac{1}{2}q+\frac{5}{6})\Gamma(\frac{1}{2}q+\frac{7}{6})}\,dq,$$

$$x_2(w,0) = \frac{1}{2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{3q/2-1/2}\tan\tfrac{1}{2}\pi q\,\frac{\cos\pi q+\cos\frac{1}{3}\pi}{\cos\pi q-\cos\frac{1}{3}\pi}\times$$
$$\times\frac{\Gamma(\frac{1}{2}q-\frac{1}{3})\Gamma(\frac{1}{2}q+\frac{1}{3})}{\Gamma(\frac{1}{2}q+\frac{1}{6})\Gamma(\frac{1}{2}q+\frac{5}{6})}\,dq, \quad (-1 < c < \tfrac{1}{3}). \tag{3.9}$$

The last integrand has simple poles at

$$q = \pm(2n+1), \quad 2n+\tfrac{1}{3}, \quad 2n+\tfrac{5}{3}, \qquad \text{where } n = 0, 1, 2, \ldots,$$

and for $w \gtrless 1$ we may use large semi-circles in $\mathscr{R}q \lessgtr 0$, respectively. Then for $w > 1$,

$$\left.\begin{aligned} x_2(w,0) &= \frac{1}{\pi}\sum_{n=0}^{\infty} a_n w^{-3n-2}, \\ \text{where}\quad a_n &= \frac{\Gamma(n+\frac{4}{3})\Gamma(n+\frac{2}{3})}{\Gamma(n+\frac{7}{6})\Gamma(n+\frac{11}{6})} \sim \frac{1}{n}-\frac{3}{4n^2}+\ldots \quad \text{for } n\to\infty, \\ a_0 &= \tfrac{4}{5}\sqrt{3}; \end{aligned}\right\} \tag{3.10 a}$$

and for $w < 1$,

$$\left.\begin{aligned} x_2(w,0) &= \frac{1}{3\pi}\sum_{n=0}^{\infty}(b_n+c_n w+d_n w^2)w^{3n}, \\ \text{where}\quad b_n &= \frac{\Gamma(n-\frac{1}{6})\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{1}{3})\Gamma(n+1)} \sim \frac{1}{n}+\frac{1}{12n^2}+\ldots \quad \text{for } n\to\infty, \\ c_n &= \frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{\Gamma(n+\frac{2}{3})\Gamma(n+\frac{4}{3})} \sim \frac{1}{n}-\frac{1}{4n^2}+\ldots, \\ d_n &= \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{7}{6})}{\Gamma(n+1)\Gamma(n+\frac{5}{3})} \sim \frac{1}{n}-\frac{7}{12n^2}+\ldots. \end{aligned}\right\} \tag{3.10 b}$$

The asymptotic forms of the coefficients show that these series (which could also be written as *generalized* hypergeometric functions) converge rapidly except near $w = 1$, and that they diverge at $w = 1$. However, the singular part of the function is easily recognized and removed from the series. Equation (3.10 a) may be written

$$\pi w^2 x_2(w,0) = -\log(1-w^{-3}) - \tfrac{3}{4}[(1-w^{-3})\log(1-w^{-3})+w^{-3}] +$$
$$+\sum_{n=0}^{\infty}\left(a_n-\frac{1}{n}+\frac{3}{4n(n-1)}\right)w^{-3n}, \tag{3.11}$$

where it is to be understood that the terms $-(1/n)$ and $3/\{4n(n-1)\}$ are to be included only for $n \geqslant 1$ and for $n \geqslant 2$, respectively. Equation (3.11) displays explicitly the logarithmic singularity of x (which corresponds to an exponential decay of the excess velocity in the duct), and the new series converges at $w = 1$ like $\sum n^{-3}$. Equation (3.10 b) can be put into a similar form.

It remains to find the constant μ, which determines the scale of the far-field solution χ_1: to this end we consider the flow at E $(w, v \to \infty)$. A suitable series for χ_2, as $v \to \infty$, is obtained by expanding $f(\lambda)$ in ascending powers of its argument:

$$\chi_2 = \frac{\sqrt{3}}{\pi}\zeta^{1/3}\int_0^\infty e^{-\lambda v}K_{1/3}(\lambda\zeta)\left[\left(\frac{2}{3}\right)^{2/3}\frac{2\sqrt{3}}{5\pi} - \left(\frac{2}{3}\right)^{2/3}\frac{8\sqrt{3}}{5\,.\,7\,.\,11\,.\,\pi}\lambda^2 + \ldots\right]d\lambda. \quad (3.12)$$

Applying the condition

$$\chi_2 + \chi_1 \sim o(\chi_1) \quad \text{for } v \to \infty,$$

which follows from the requirement of finite drag, and using (3.3 a) for χ_1, we obtain

$$\mu = \left(\frac{2}{3}\right)^{2/3}\frac{2\sqrt{3}}{5\pi}. \quad (3.13)$$

The asymptotic value of the complete solution near the edge E is now given by the second term of (3.12), and may be written

$$\chi_E = \left(\frac{2}{3}\right)^{2/3}\frac{8\sqrt{3}}{5\,.\,7\,.\,11\,.\,\pi}\frac{\partial^2}{\partial v^2}\left(\frac{\chi_1}{\mu}\right). \quad (3.14)$$

This is an elementary function which can be found from equations (3.3 b) to (3.3 d).

A more revealing (but less rigorous) method of evaluating μ is the following. The condition at the stagnation point S may be written

$$x = 0 \quad \text{at} \quad w = \delta^{-2/3}\Gamma^{1/3}, \qquad v = 0.$$

Since w is large, the leading term of an asymptotic series for $x_2(w, 0)$ suffices, and this is given by the first term of (3.10 a). Thus we require

$$x_1 + x_2 \sim -\mu 2\left(\frac{3}{2}\right)^{2/3} w^{-2} + \frac{4\sqrt{3}}{5\pi}w^{-2} = 0 \quad \text{at} \quad w = \delta^{-2/3}\Gamma^{1/3}.$$

In other words, as $w \to \infty$ on $v = 0$, x should $\to 0$ as rapidly as possible, in order that the stagnation point of the exact hodograph be simulated as closely as possible. This requirement leads to the same value of μ as before.

4. The far field and the edge field in the supersonic region

The far-field potential χ_1 and the edge-field potential χ_E are both similarity solutions up to the limiting Mach wave. Barish & Guderley (1953) have demonstrated that when the initial solution is a similarity one, the flow continues analytically to the shock wave, which is itself a similarity curve, and that the solution behind the shock, while a different function, has the same similarity as that before the shock. However, Barish &

Guderley, who studied plane and axi-symmetric flows simultaneously, used a non-linear differential equation associated with the flow in the physical plane, and had to integrate this equation numerically to obtain their results.

In this paper we prefer to work in the hodograph plane, where our asymptotic solutions are elementary functions, both before and after the shock. The only numerical work required will be to find the roots of certain algebraic equations, which provide the constants in the solutions.

In place of α we introduce the variable

$$\xi = \sin^{-1}(1-\alpha)^{1/2} = \sin^{-1}\frac{(\zeta^2-v^2)^{1/2}}{\zeta},$$

which increases from 0 to π as v decreases from ζ to $-\zeta$. The upstream solutions are both of form

$$\left.\begin{aligned} &\chi = u^{-n}fn(\xi), && (n = \tfrac{3}{2}m),\\ &x = u^{-n-1}k_a f_a(\xi), && y = u^{-n-3/2}k_a g_a(\xi), \end{aligned}\right\} \tag{4.1}$$

and the solutions downstream of the shock will be written

$$x = u^{-n-1}k_b f_b(\eta), \qquad y = u^{-n-3/2}k_b g_b(\eta), \tag{4.2}$$

where

$$\eta = \xi - \tfrac{1}{2}\pi.$$

f_b and g_b are determined by the similarity index n, and by the boundary condition $y = 0$ on $\eta = 0$: they are, in fact, hypergeometric functions which again reduce to elementary form.

For the far field, by (3.3 d):

$$\left.\begin{aligned} n &= 1, \qquad k_a = \frac{2\sqrt{3}}{5\pi},\\ f_a(\xi) &= \frac{2\sin\frac{1}{3}\xi}{\sin\xi} - \cos\xi\, g_a(\xi),\\ g_a(\xi) &= \frac{-\sin\frac{4}{3}\xi + 2\sin\frac{2}{3}\xi}{\sin^3\xi},\\ f_b(\eta) &= \frac{2\cos\frac{1}{3}\eta}{\cos\eta} + \sin\eta\, g_b(\eta),\\ g_b(\eta) &= \frac{\sin\frac{4}{3}\eta + 2\sin\frac{2}{3}\eta}{\cos^3\eta}, \end{aligned}\right\} \tag{4.3}$$

and for the edge field, by (3.14):

$$\left.\begin{aligned} n &= 4, \qquad k_a = \frac{\sqrt{3}}{5.7.11.\pi},\\ f_a(\xi) &= \frac{8(5\sin\frac{7}{3}\xi - 14\sin\frac{5}{3}\xi + 35\sin\frac{1}{3}\xi)}{\sin^5\xi} - \cos\xi\, g_a(\xi),\\ g_a(\xi) &= \frac{10(-2\sin\frac{10}{3}\xi + 7\sin\frac{8}{3}\xi - 30\sin\frac{4}{3}\xi + 42\sin\frac{2}{3}\xi)}{\sin^7\xi},\\ f_b(\eta) &= \frac{8(-5\cos\frac{7}{3}\eta - 14\cos\frac{5}{3}\eta + 35\cos\frac{1}{3}\eta)}{\cos^5\eta} + \sin\eta\, g_b(\eta),\\ g_b(\eta) &= \frac{10(-2\sin\frac{10}{3}\eta - 7\sin\frac{8}{3}\eta + 30\sin\frac{4}{3}\eta + 42\sin\frac{2}{3}\eta)}{\cos^7\eta}. \end{aligned}\right\} \tag{4.4}$$

Let suffices $(\ )_1$ and $(\ )_2$ denote, for the moment, conditions immediately before and after the shock, let $u_2/u_1 = \sigma$, and let $k_b/k_a = \kappa$. The following conditions apply.

(i) x and y must be continuous across the shock:

$$\frac{f_b(\eta_2)}{f_a(\xi_1)} = \frac{\sigma^{n+1}}{\kappa}, \qquad \frac{g_b(\eta_2)}{g_a(\xi_1)} = \frac{\sigma^{n+3/2}}{\kappa}. \tag{4.5 a, b}$$

(ii) The jump in (u, v) must be along the transonic shock polar:

$$v_2 - v_1 = (u_1 - u_2)\left(\frac{u_1 + u_2}{2}\right)^{1/2},$$

i.e.

$$\frac{2}{3}(-\sigma^{3/2}\sin\eta_2 - \cos\xi_1) - (1-\sigma)\left(\frac{1+\sigma}{2}\right)^{1/2} = 0. \tag{4.6}$$

(iii) The shock slope must be normal to the vectorial velocity jump:

$$\frac{dx_1}{dy_1} = \frac{dx_2}{dy_2} = \frac{v_2 - v_1}{u_1 - u_2} = \left(\frac{u_1 + u_2}{2}\right)^{1/2},$$

i.e.

$$\frac{n+1}{n+\frac{3}{2}}\frac{f_a(\xi_1)}{g_a(\xi_1)} = \left(\frac{\sigma+1}{2}\right)^{1/2}, \qquad \frac{n+1}{n+\frac{3}{2}}\frac{f_b(\eta_2)}{g_b(\eta_2)} = \left(\frac{1+\sigma}{2\sigma}\right)^{1/2}. \tag{4.7 a, b}$$

These are four, rather than five, independent equations, since (4.5 a) and (4.5 b) combine to

$$\frac{f_b(\eta_2)g_a(\xi_1)}{f_a(\xi_1)g_b(\eta_2)} = \sigma^{-1/2},$$

and so do (4.7 a) and (4.7 b). The unknowns are ξ_1, η_2, σ, and κ; and the solution of these equations is not difficult. From (4.7) one can tabulate or plot ξ_1 and η_2 as functions of σ. The left-hand side of (4.6) is then a known function of σ, and one seeks its zero. There is only one such zero within the physically possible range of the variables. The results of this calculation are shown in the following table: here c is the constant in the equation of the shock wave $y = cx^{(2n+3)/(2n+2)}$.

	n	σ	ξ_1	η_2	κ	c
Far field	1	0·345	154·0°	27·2°	1·124	0·616
Edge field	4	0·735	126·7°	21·2°	1·004	1·430

5. The pressure distribution along the outer wall

The edge field and the far field are now completely determined, but the flow in the intermediate region is known only up to the shock, whose position is also not fully known. Strictly, the solution should be completed by means of the numerical method of characteristics, but in the present case the following empirical procedure was though to be sufficient.

There are two lines $u > 0$, $v = 0$ in the problem: the first lies between the limiting Mach wave and the shock, and on the first sheet of the hodograph plane; the second is the outer wall of the intake and lies on the second sheet of the hodograph. The first of these lines would probably be used as a starting point of the characteristics computation (cf. Guderley 1954). Let the values of x on these lines be denoted by $x_a(u,0)$ and $x_b(u,0)$, respectively. Both these functions are monotonic decreasing and might be expected to be qualitatively similar, since both represent a transition from the edge field to the far field. Now the foregoing results show that

$$\left.\begin{aligned} \frac{x_b(u,0)}{x_a(u,0)} &= 2{\cdot}248[1+O(u^2)] \quad \text{for } u\to 0, \\ \text{and} \qquad \frac{x_b(u,0)}{x_a(u,0)} &= 2{\cdot}008[1+O(u^{-3})] \quad \text{for } u\to\infty. \end{aligned}\right\} \tag{5.1}$$

Accordingly it has been assumed here that

$$\frac{x_b(u,0)}{x_a(u,0)} \sim 2 \quad \text{for all } u. \tag{5.2}$$

$x_a(u,0)$ is easily calculated by the method already used for $x(w,0)$. We have

$$\begin{aligned} x_{a1}(u,0) &= \frac{2\sqrt{3}}{5\pi}u^{-2}, \\ x_{a2}(u,0) &= -\frac{\sqrt{3}}{\pi}\frac{\partial}{\partial u}\left\{\zeta^{1/3}\int_0^\infty K_{1/3}(\lambda\zeta)f(\lambda)\frac{d\lambda}{\lambda}\right\}, \\ &= \frac{1}{4}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u^{3q/2-1/2}\sec\tfrac{1}{2}\pi q\,\frac{\cos\pi q+\cos\frac{1}{3}\pi}{\cos\pi q-\cos\frac{1}{3}\pi}\times \\ &\qquad\times\frac{\Gamma(\frac{1}{2}q-\frac{1}{3})\Gamma(\frac{1}{2}q+\frac{1}{3})}{\Gamma(\frac{1}{2}q+\frac{1}{6})\Gamma(\frac{1}{2}q+\frac{5}{6})}\,dq, \qquad (-1<c<\tfrac{1}{3}), \\ &= -\frac{1}{2\pi}\sum_{n=0}^{\infty}a_n(-u)^{-3n-2}, \qquad (u\geqslant 1), \\ &= \frac{1}{3\pi}\sum_{n=0}^{\infty}(b_n+\tfrac{1}{2}c_n u+d_n u^2)(-u)^{3n}, \qquad (u\leqslant 1), \end{aligned} \tag{5.3}$$

where the coefficients are given in (3.10). These are alternating series, which converge at $u=1$, but the convergence is improved by introducing terms in $\log(1+u^{-3})$ and $\log(1+u^3)$, as before.

6. The drag

In view of the singularity at the edge E, which makes our approximate solution locally invalid, the intake drag (as defined in § 1) must be calculated from the momentum flux across some suitable curve, such as BGO_2 in figure 3. (In this figure, O_1, O_2, I_1, I_2 represent points which are actually at infinity.) The general drag integral of transonic theory is derived in Appendix III; in the present case it becomes

$$\frac{1}{\delta^2}C_D = \frac{1}{\delta^2}\frac{\frac{1}{2}D}{\frac{1}{2}\rho_0 U_0^2 L} = \int_{BGO_2}\{2uv\,dx+(v^2+\tfrac{2}{3}u^3-\tfrac{1}{6}[u]^3)\,dy\}, \tag{6.1}$$

where $[u]$ denotes the velocity jump across the shock wave (downstream value minus upstream value), and is a function of y only. The term in $[u]^3$ accounts for the change in entropy and stagnation pressure across the shock, and is to be included only at points on BGO_2 behind the shock. The integral (6.1) is invariant under the changes of the path (provided that the termini remain at some point on the outer intake wall and at O_2), and the path may therefore be deformed to $BE_1E_2I_1I_2O_1O_2$. Then the integral around the small contour E_1E_2 gives the edge force, and that along I_1I_2 gives the pre-entry drag: the other parts of the path make no contribution.

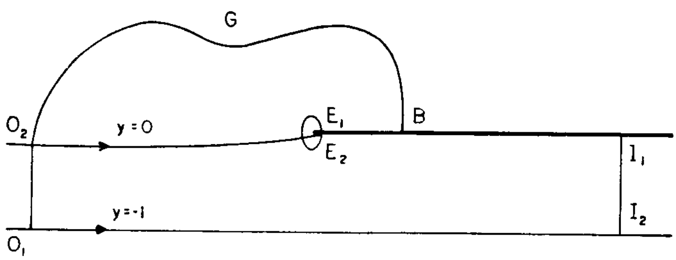

Figure 3.

Now near E the flow is represented asymptotically by a similarity solution, $\chi = u^{-n}fn(\alpha)$, and the corresponding drag integral is

$$D_{E_1E_2} \sim O(u^{-n+3/2}) \sim O(y^{(2n-3)/(2n+3)}) \tag{6.2}$$

(details of this expression are given in Appendix III).

It follows that (i) the far-field solution χ_1 must be removed in the neighbourhood of the edge ($x, y \to 0$; $u, v \to \infty$) because it would lead to an infinite force, and (ii) the true solution yields zero edge force. For χ_1 is the similarity solution with $n = 1$, so that the corresponding edge-force integral would $\to \infty$ as $u \to \infty$. (Since the geometry behind the limiting Mach wave may be varied without affecting the flow upstream, there is no general possibility of a cancellation between terms tending to infinity in the edge-force integral.) On the other hand the true edge flow is represented by the similarity solution with $n = 4$, so that the true edge-force integral $\to 0$ as $u \to \infty$.

The pre-entry drag is easily calculated from (6.1). On I_1I_2 this reduces to

$$\frac{1}{\delta^2} C_{D_{\text{pre}}} = \int_0^{-1} \tfrac{2}{3}u^3\, dy = \tfrac{2}{3}. \tag{6.3}$$

It remains to find the cowl drag of the slender-wedge intakes discussed in § 1. Let the slope of the wedge face be $\epsilon\delta$, and let its length be $lL(\delta\Gamma)^{1/3}$, where l is $O(1)$. For $\epsilon \to 0$ the pressure changes induced by the wedge are $O(\epsilon\delta)$, and their contribution to the drag is therefore small compared with that of the flat-plate pressures. Hence, by (6.1)

$$\frac{1}{\delta^2} C_{D_{\text{cowl}}} \sim -2\epsilon \int_0^l u\, dx,$$

where u refers to values on the outside wall of the flat-plate intake. Numerical values are given in the next section.

7. Results

Pressure distributions along the centre-line, $y = -1$, along the inner wall $y = 0-$, and along the outer wall, $y = 0+$, are shown in figure 4.

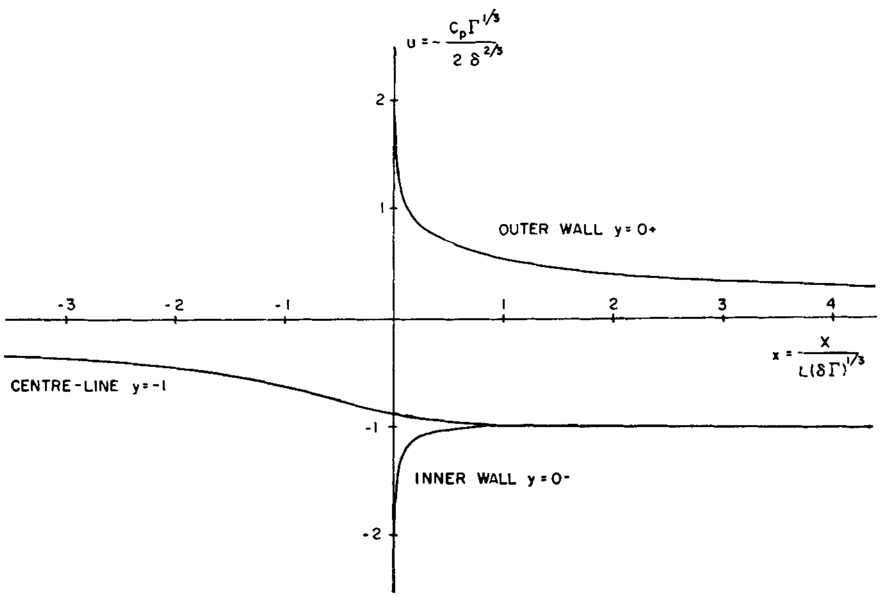

Figure 4.

This picture is supplemented by the following asymptotic approximations (which, for the external wall, are based on (5.1) rather than on the approximation (5.2))

$$\begin{aligned}
u(x, -1) &\sim -0{\cdot}664(-x)^{-1/2} && \text{for } x \to -\infty,\\
u(x, 0+) &\sim 0{\cdot}704x^{-1/2} && \text{for } x \to \infty,\\
u(x, 0+) &\sim 0{\cdot}713x^{-1/5} && \text{for } x \to 0,\\
u(x, 0-) &\sim -0{\cdot}712x^{-1/5} && \text{for } x \to 0,\\
u(x, 0-) &\sim -1-0{\cdot}147e^{-\pi x} && \text{for } x \to \infty,\\
u(x, -1) &\sim -1+0{\cdot}147e^{-\pi x} && \text{for } x \to \infty.
\end{aligned}$$

It is evident that the external decay of disturbances before and after the inlet station is slow (although it must be remembered that x is a 'stretched' coordinate), and that the edge singularity is very weak. It is also striking that over 90% of the compression of the entering fluid takes place ahead of the inlet station, and that the decay of disturbances within the duct is extremely rapid, so that one-dimensional theory provides a very good approximation to the flow within the duct.

As a basis for comparison, we quote some results for the present intake in an incompressible stream. In this case the reduced variables are given by

$$\delta = \frac{U_0 - U_1}{U_0}, \quad X = Lx, \quad Y = Ly, \quad U' = \delta u(x,y), \quad V' = \delta v(x,y),$$

and the small-disturbance solution is

$$x + iy = \frac{1}{\pi}\left(\frac{1}{u-iv} + \log\frac{u-iv}{u-iv+1}\right),$$

so that

$$\begin{aligned} u(x,-1) &\sim 0{\cdot}318x^{-1} && \text{for } x \to -\infty, \\ u(x,0+) &\sim 0{\cdot}318x^{-1} && \text{for } x \to \infty, \\ u(x,0\pm) &\sim \pm 0{\cdot}399x^{-1/2} && \text{for } x \to 0, \\ u(x,0-) &\sim -1-0{\cdot}368e^{-\pi x} && \text{for } x \to \infty, \\ u(x,-1) &\sim -1+0{\cdot}368e^{-\pi x} && \text{for } x \to \infty. \end{aligned}$$

Within the framework of reduced coordinates, the external disturbances of the incompressible flow are therefore more concentrated near the inlet station than are those of the transonic flow: the extent to which this effect is reversed, when physical coordinates are used, depends upon the magnitude of δ.

The pre-entry drag of the intake in transonic flow is

$$C_{D_{\text{pre}}} = \frac{D_{\text{pre}}}{\frac{1}{2}\rho_0 U_0^2 \cdot 2L} = \frac{2}{3}\delta^2.$$

Slender-wedge cowls, whose wedge face has slope $\epsilon\delta$, ($\epsilon \ll 1$), and length $lL(\delta\Gamma)^{1/3}$, have the following drag:

l	$\frac{1}{2}$	1	$\frac{3}{2}$
$\frac{1}{\delta^2\epsilon} C_{D_{\text{cowl}}}$	$-0{\cdot}9$	$-1{\cdot}5$	$-2{\cdot}0$

The decrease in drag with increasing frontal area is therefore quite rapid, at least initially.

I am indebted to Professor P. Germain and to Professor J. D. Cole for a number of most helpful discussions, and to Mrs A. Tingley for help with the numerical computations. This work was completed while I was on leave of absence at the Guggenheim Aeronautical Laboratory, California Institute of Technology, and was partly sponsored by the U.S. Air Force Office of Scientific Research, under Contract AF-18(600)-383.

Appendix I. Elementary solutions of the Tricomi equation

By separation of variables one finds the following elementary solutions of equation (2.5):

$$\left.\begin{aligned} e^{\pm\lambda v}\zeta^{1/3}K_{1/3}(\lambda\zeta) &= \frac{\pi}{\sqrt{3}}e^{\pm\lambda v}z^{1/3}[J_{1/3}(\lambda z)+J_{-1/3}(\lambda z)], \\ \text{and}\qquad e^{\pm\lambda v}\zeta^{1/3}I_{1/3}(\lambda\zeta) &= -e^{\pm\lambda v}z^{1/3}J_{1/3}(\lambda z). \end{aligned}\right\} \tag{I.1}$$

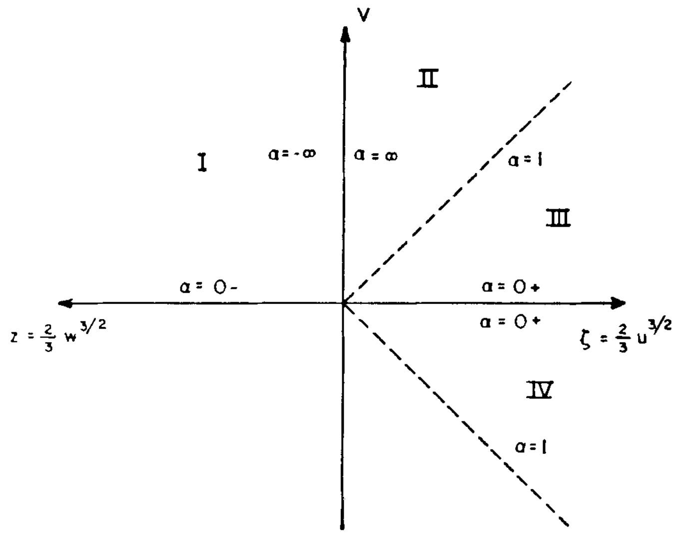

Figure 5.

Alternatively, if one seeks similarity solutions of form $\chi = \zeta^{-m}fn(\alpha)$, where $\alpha = v^2/\zeta^2 = 9v^2/4u^3$, these turn out to be, in a neighbourhood of the characteristic $v = \zeta$ (see figure 5),

$$\left.\begin{aligned} &\zeta^{-m}F\left(\frac{1}{2}m,\quad \frac{1}{2}m+\frac{1}{3};\quad m+\frac{5}{6};\quad 1-\alpha\right), \\ &\text{and} \\ &\zeta^{-m}(\alpha-1)^{-m+1/6}F\left(-\frac{1}{2}m+\frac{1}{6},\quad -\frac{1}{2}m+\frac{1}{2};\quad -m+\frac{7}{6};\quad 1-\alpha\right), \end{aligned}\right\} \tag{I.2}$$

with corresponding hypergeometric functions applying elsewhere in the field. The singularities of the hypergeometric equation, $\alpha = 0, \infty, 1$, correspond, respectively, to the line $v = 0$, to the sonic line, and to the important characteristics $\alpha = 1$. As Guderley (1948) has demonstrated, one of these two characteristics is generally the limiting Mach wave, which divides both the physical and the hodograph fields into (i) the subsonic region, together with an initial part of the supersonic region which influences the subsonic flow, and (ii) that part of the supersonic region which does not influence the upstream field.

Most analytic solutions of problems in transonic flow have been obtained by superposition of the solutions (I.1) and (I.2), but the relatively simple relation between these forms does not appear to have been emphasized in the literature. With a view to the present application, we discuss this relation here. Consider

$$\chi = \zeta^{1/3} \int_0^\infty e^{-\lambda v}[g(\lambda)K_{1/3}(\lambda\zeta) + h(\lambda)I_{1/3}(\lambda\zeta)]\frac{d\lambda}{\lambda}, \tag{I.3}$$

where g and h are assumed to be algebraic rather than exponential. Since for $\lambda \to \infty$

$$e^{-\lambda v}K_{1/3}(\lambda\zeta) \sim \left(\frac{\pi}{2\lambda\zeta}\right)^{1/2} e^{-\lambda(v+\zeta)},$$

$$e^{-\lambda v}I_{1/3}(\lambda\zeta) \sim \frac{1}{(2\pi\lambda\zeta)^{1/2}} e^{-\lambda(v-\zeta)},$$

the first part of (I.3) converges for $\mathscr{R}(v+\zeta) > 0$ (that is in the regions I, II, III, IV of the hodograph plane, figure 5), whereas the second converges for $\mathscr{R}(v-\zeta) > 0$ (that is, only in I and II). Thus we expect the $K_{1/3}$-solution to be associated with only the first solution in (I.2), which is analytic on $v = \zeta$. In fact by standard integrals, and by properties of the hypergeometric functions (Erdélyi 1954, 1953), one finds that

$$\zeta^{1/3} \int_0^\infty e^{-\lambda v}\lambda^{m-2/3}K_{1/3}(\lambda\zeta)\, d\lambda$$
$$= \frac{\pi^{1/2}\,\Gamma(m)\Gamma(m+\frac{2}{3})}{2^{m+1/3}\,\Gamma(m+\frac{5}{6})}\, \zeta^{-m} F\left(\frac{1}{2}m, \quad \frac{1}{2}m + \frac{1}{3}; \quad m + \frac{5}{6}; \quad 1-\alpha\right), \qquad (m > 0). \tag{I.4}$$

A similar result for the $I_{1/3}$-solution involves both the hypergeometric functions in (I.2).

To obtain a further relation we introduce the Mellin transform. The definition integral of this transform is

$$\tilde{\phi}(p) = \int_0^\infty \zeta^{p-1}\phi(\zeta)\, d\zeta, \qquad (A < \mathscr{R}p < B), \tag{I.5}$$

which we also write $\tilde{\phi}(p) \risingdotseq \phi(\zeta)$; the inversion integral is

$$\phi(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^{-p}\,\tilde{\phi}(p)\, dp, \qquad (A < c < B); \tag{I.6}$$

and the composition product is

$$\tilde{\phi}_1(p+a)\tilde{\phi}_2(-p+b) \risingdotseq \zeta^a \int_0^\infty \lambda^{a+b-1}\phi_1(\lambda\zeta)\phi_2(\lambda)\, d\lambda \tag{I.7}$$

(provided that $\tilde{\phi}_1(p+a)$ and $\tilde{\phi}_2(-p+b)$ have a common strip of convergence). Now let

$$e^{-\lambda\alpha^{1/2}}K_{1/3}(\lambda) = G(\lambda, \alpha).$$

Then $\tilde{G}(p+\frac{1}{3}, \alpha)$ is given by (I.4), with $m = p$, $v = \zeta\alpha^{1/2}$, $\zeta = 1$. Now if in (I.3) we set $h(\lambda) = 0$, and take the Mellin transform of both sides, we have, by (I.7),

$$\tilde{\chi}(p, \alpha) = \tilde{G}(p+\tfrac{1}{3}, \alpha)\tilde{g}(-p-\tfrac{1}{3}),$$

provided that $\tilde{g}(q)$ converges somewhere in $\mathscr{R}q < -\frac{1}{3}$. Hence

$$\chi(\zeta,\alpha) = \zeta^{1/3}\int_0^\infty e^{-\lambda v}K_{1/3}(\lambda\zeta)g(\lambda)\,\frac{d\lambda}{\lambda}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta^{-p}\,\frac{\pi^{1/2}\,\Gamma(p)\Gamma(p+\frac{2}{3})}{2^{p+1/3}\,\Gamma(p+\frac{5}{6})}\times$$

$$F\left(\frac{1}{2}p,\quad \frac{1}{2}p+\frac{1}{3};\quad p+\frac{5}{6};\quad 1-\alpha\right)\tilde{g}(-p-\tfrac{1}{3})\,dp, \qquad \text{(I.8)}$$

and we have transformed a superposition of the solutions (I.1) into a superposition of the solutions (I.2), the weighting function of the first form being replaced by its Mellin transform in the second. The complex integral can often be expressed as a series by contour integration.

Appendix II. Uniqueness of y_2

In this appendix the problem for y_2 is transformed into a problem of the standard Tricomi type by means of a certain inversion which leaves the equation invariant (see, for example, Germain & Bader (1952)). The $(z(\zeta), v)$-plane is mapped on to an $(\alpha(\gamma), \beta)$-plane by the transformation

$$\sqrt{\{(v+1)^2+z^2\}} \equiv \sqrt{\{(v+1)^2-\zeta^2\}} \equiv r = \frac{1}{\rho} \equiv \frac{1}{\sqrt{(\beta^2+\alpha^2)}} \equiv \frac{1}{\sqrt{(\beta^2-\gamma^2)}},$$

$$\frac{v+1}{\sqrt{\{(v+1)^2+z^2\}}} \equiv \frac{v+1}{\sqrt{\{(v+1)^2-\zeta^2\}}} \equiv t = \tau \equiv \frac{\beta}{\sqrt{(\beta^2+\alpha^2)}} \equiv \frac{\beta}{\sqrt{(\beta^2-\gamma^2)}},$$

$$y(r,t) = \rho^{1/3}\eta(\rho,\tau).$$

The line $v = 0$, z real, maps on to the semi-circle $\rho = \tau$, or $\alpha^2+(\beta-\frac{1}{2})^2 = \frac{1}{4}$, $0 \leqslant \tau \leqslant 1$; the quarter-circle $r \to \infty$, $0 \leqslant t \leqslant 1$, maps on to $\rho \to 0$, $0 \leqslant \tau \leqslant 1$; the characteristic at infinity $v+\zeta = k$, $k \to \infty$, maps on to $\beta-\gamma = 0$, $\gamma \leqslant \frac{1}{2}$; and the characteristic $v-\zeta = 0$ maps on to $\beta+\gamma = 1$, $\gamma \leqslant \frac{1}{2}$. The problem for y_2, namely

$$\text{(i)}\ y_{rr} + \frac{1-t^2}{r^2}y_{tt} + \frac{4}{3r}\left(y_r - \frac{t}{r}y_t\right) = 0,$$

$$\text{(ii)}\ y = -1 \quad \text{on} \quad v = 0,\ \tfrac{2}{3} > z \geqslant 0,$$
$$= 0 \quad \text{on} \quad v = 0,\ z > \tfrac{2}{3},$$

$$\text{(iii)}\ y \sim O(r^{-5/3}) \quad \text{for } z \text{ real},\ r \to \infty,$$
$$\sim O(v^{-5/3}) \quad \text{for } \zeta \text{ real},\ v \to \infty,$$

then becomes

$$\text{(i)}\ \eta_{\rho\rho} + \frac{1-\tau^2}{\rho^2}\eta_{\tau\tau} + \frac{4}{3\rho}\left(\eta_\rho - \frac{\tau}{\rho}\eta_\tau\right) = 0,$$

$$\text{(ii)}\ \eta = -\rho^{-1/3} \quad \text{on} \quad \rho = \tau, \quad \frac{3}{\sqrt{13}} < \tau \leqslant 1,$$

$$= 0 \quad \text{on} \quad \rho = \tau, \quad 0 \leqslant \tau < \frac{3}{\sqrt{13}},$$

$$\text{(iii)}\ \eta = 0 \quad \text{on} \quad \rho = 0, \quad \gamma \leqslant \tfrac{1}{2}.$$

The uniqueness of such a problem, in which data are prescribed on an arc in the elliptic half-plane and on an adjacent characteristic in the hyperbolic half-plane, was established by Tricomi (1923).

Appendix III. The drag integral of transonic theory

In this appendix we first develop the general drag integral of transonic theory: this involves no more labour than the derivation of the particular form which is of interest here. Let ϕ be the perturbation velocity potential of a three-dimensional transonic flow, such that the total velocity at any point is $U_0(1+\nabla\phi)$. The governing differential equation is

$$(1-M_0^2-\Gamma\phi_X)\phi_{XX}+\phi_{YY}+\phi_{ZZ}=0. \tag{III.1}$$

Consider the drag on a portion A of some solid boundary, and let S be a surface in the fluid such that $S+A$ form a closed surface. Then by the equations of mass and momentum

$$D = \iint_A N_{(X)}(P-P_0)\,dA, \tag{III.2a}$$

$$= -\iint_S [N_{(X)}(P-P_0+\rho U_0^2\phi_X)+\rho U_0^2\phi_X(\mathbf{N}\,.\,\nabla\phi)]\,dS, \tag{III.2b}$$

where $\mathbf{N}$ is the normal outward from $S+A$, $N_{(X)}$ is its X-component, and P and ρ denote the pressure and density. If $P-P_0$ is bounded on A, (III.2a) and (III.2b) are wholly equivalent, but if there is a singularity on A, an approximate solution will not be physically valid in its neighbourhood. We assume, however, that the solution is valid in the field away from the singularity: then the drag is defined by (III.2b).

By expansion of the usual equations of inviscid, homenergic flow, we find that

$$\frac{P-P_0}{\frac{1}{2}\rho_0 U_0^2} = -2\phi_X-(\nabla\phi)^2+M_0^2\phi_X^2+M_0^2\phi_X(\nabla\phi)^2-\tfrac{1}{3}(2-\gamma)M_0^4\phi_X^3+ \\ +\tfrac{1}{6}(\gamma+1)M_0^4\textstyle\sum[\phi_X]^3+O\{(\nabla\phi)^4\}, \tag{III.3a}$$

$$\frac{\rho}{\rho_0} = 1-M_0^2\phi_X-\tfrac{1}{2}M_0^2(\nabla\phi)^2+\tfrac{1}{2}(2-\gamma)M_0^4\phi_X^2+O\{(\nabla\phi)^3\}. \tag{III.3b}$$

Here $[\phi_X]$ denotes the velocity jump across a shock (downstream value minus upstream value) and the term in $[\phi_X]^3$ accounts for the change in stagnation pressure across any shock waves upstream of the point in question*. To our order of accuracy $[\phi_X]$ is a function of Y and Z only. If these expressions are now inserted in the drag integral (III.2b), there results

$$\frac{D}{\frac{1}{2}\rho_0 U_0^2} = \iint_S \{N_{(X)}((\nabla\phi)^2+M_0^2\phi_X^2-\tfrac{2}{3}(2-\gamma)M_0^4\phi_X^3-\tfrac{1}{6}(\gamma+1)M_0^4\textstyle\sum[\phi_X]^3)- \\ -2\mathbf{N}\,.\,\nabla\phi(\phi_X-M_0^2\phi_X^2)\}\,dS. \tag{III.4}$$

* The need for this term was brought to my attention by Professor Germain and Professor Cole.

This is the drag integral of second-order compressible flow theory. For transonic flow we may make the further approximations

$$M_0^4\phi_X^3 \sim M_0^2\phi_X^3,$$
$$(\mathbf{N}\,.\,\nabla\phi)\phi_X^2 \sim N_{(X)}\phi_X^3,$$

to obtain

$$\frac{D}{\frac{1}{2}\rho_0 U_0^2} = \iint_S \{N_{(X)}((\nabla\phi)^2+M_0^2\phi_X^2+\tfrac{2}{3}\Gamma\phi_X^3-\tfrac{1}{6}\Gamma\sum[\phi_X]^3)-2\mathbf{N}\,.\,\nabla\phi\,\phi_X\}\,dS. \tag{III.5}$$

It is readily verified, by means of Gauss's theorem, the differential equation (III.1), and the conservation equations for a shock wave, that this drag integral is invariant under changes of the surface S (provided that $S+A$ remains a closed surface and that there are no singularities between different surfaces S).

For plane flows with $M_0 = 1$, (III.5) reduces to

$$\frac{1}{\delta^2}C_D = \frac{1}{\delta^2}\frac{D}{\frac{1}{2}\rho_0 U_0^2 L} = \int_C \{2uv\,dx+(v^2+\tfrac{2}{3}u^3-\tfrac{1}{6}\sum[u]^3)\,dy\}, \tag{III.6}$$

where the integral is taken counter-clockwise in the physical plane, and where the reduced variables of (2.2) have been introduced.

Next, following Germain (1957), we consider the form of the drag integral when the flow may be represented on C by a similarity solution of the form considered in (I.2); that is, when

$$\chi = u^{-n}h(\alpha), \qquad (n=\tfrac{3}{2}m),$$

so that h is some solution of the hypergeometric equation

$$\alpha(1-\alpha)h''+[\tfrac{1}{2}-(\tfrac{4}{3}+\tfrac{2}{3}n)\alpha]h'-\tfrac{1}{9}n(1+n)h = 0. \tag{III.7}$$

Equation (III.6) now becomes

$$\begin{aligned}\frac{1}{\delta^2}C_D = \int_{C'} \{&u^{-n+1/2}[((\tfrac{8}{3}n+2)\alpha^{3/2}-(2n+3)\alpha^{1/2})h'+\tfrac{4}{3}(n^2+n)\alpha^{1/2}h]\,du+\\ &+u^{-n+3/2}[(-\tfrac{8}{3}\alpha^{3/2}+2\alpha^{1/2})h''-((\tfrac{4}{3}n+\tfrac{10}{3})\alpha^{1/2}-\alpha^{-1/2})h']\,d\alpha\}-\\ &-\frac{1}{6}\int_C \sum[u]^3\,dy,\end{aligned} \tag{III.8}$$

where C' is the image of C in the hodograph plane. Now the invariance property of the drag integral ensures that the integrand in (III.8) is an exact differential, and this may also be verified by means of (III.7). Hence it follows from the du-term that

$$\begin{aligned}\frac{1}{\delta^2}C_D = &\left[\frac{u^{-n+3/2}}{-n+\frac{3}{2}}\{((\tfrac{8}{3}n+2)\alpha^{3/2}-(2n+3)\alpha^{1/2})h'+\tfrac{4}{3}(n^2+n)\alpha^{1/2}h\}\right]_{C'}-\\ &-\frac{1}{6}\int_C \sum[u]^3\,dy.\end{aligned} \tag{III.9}$$

In general C and C' cross shock waves, and the integrated term in (III.9) is then only piecewise continuous: its discontinuities must be taken into account when (III.9) is evaluated.

Germain (1957) has pointed out that if C is a large contour in the physical plane with termini on $y = 0$, $v = 0$, the appropriate solution is that with $n = 1$, so that the integrated term in (III.9) vanishes in the limit. The drag is then seen to be entirely due to the entropy gain across shock waves.

Note added in proof

It is clear from (III.8) and (III.9) that an edge force is possible, in transonic flow with $M_0 = 1$, only if $n = \frac{5}{2}$. Such a flow has been studied by Nonweiler (1958), who finds that the solution

$$\chi = \text{const. } w^{-3/2}(1-\alpha)^{-5/6} \qquad (w \geqslant 0)$$

corresponds to the flow past the body $Y \propto X^{2/5}$, the velocity on this body being sonic for $X > 0$. Nonweiler calculates the edge force by integrating surface pressure on the related bodies $Y \propto X^{\nu}$ and letting $\nu \to \frac{2}{5}$. The same result is readily obtained from (III.8), in which the du-term and the shock term vanish. This shows that the only assumption implicit in Nonweiler's result is that the velocity in the field away from the singularity is given correctly to the lowest order.

References

Barish, D. T. & Guderley, G. 1953 Asymptotic forms of shock waves in flows over symmetrical bodies at Mach one, *J. Aero. Sci.* **20**, 491.

Erdélyi, A. (Ed.) 1953 *Higher Transcendental Functions*, vol. 1. New York : McGraw-Hill.

Erdélyi, A. (Ed.) 1954 *Tables of Integral Transforms*, vol. 1. New York : McGraw-Hill.

Frankl, F. I. 1947 An investigation of the theory of a wing of infinite span moving at the speed of sound, *Doklady Akad. Nauk. SSSR* (N.S.) **57**, 661.

Germain, P. & Bader, R. 1952 Sur quelques problèmes relatifs a l'équation de type mixte de Tricomi, *O.N.E.R.A. Publication* no. 54.

Germain, P. 1957 Lectures on transonic flow at the California Institute of Technology.

Guderley, G. 1948 Singularities at the sonic velocity, *Wright-Patterson Air Force Base Tech. Rep.* no. F–TR–1171–ND.

Guderley, G. 1954 The flow over a flat plate with a small angle of attack at Mach number one, *J. Aero. Sci.* **21**, 261.

Nonweiler, T. R. F. 1958 The sonic flow about some symmetric half-bodies, *J. Fluid Mech.* **4**, 140.

Spreiter, J. R. 1954 On alternative forms of the basic equations of transonic flow theory, *J. Aero. Sci.* **21**, 70.

Tricomi, F. 1923 Sulle equazioni lineari alle derivate parziali di 2° ordine di tipo misto, *Mem. Real. Acad. Lincei; classe Fis. Mat. e. Nat.*, **14** (5), 133.